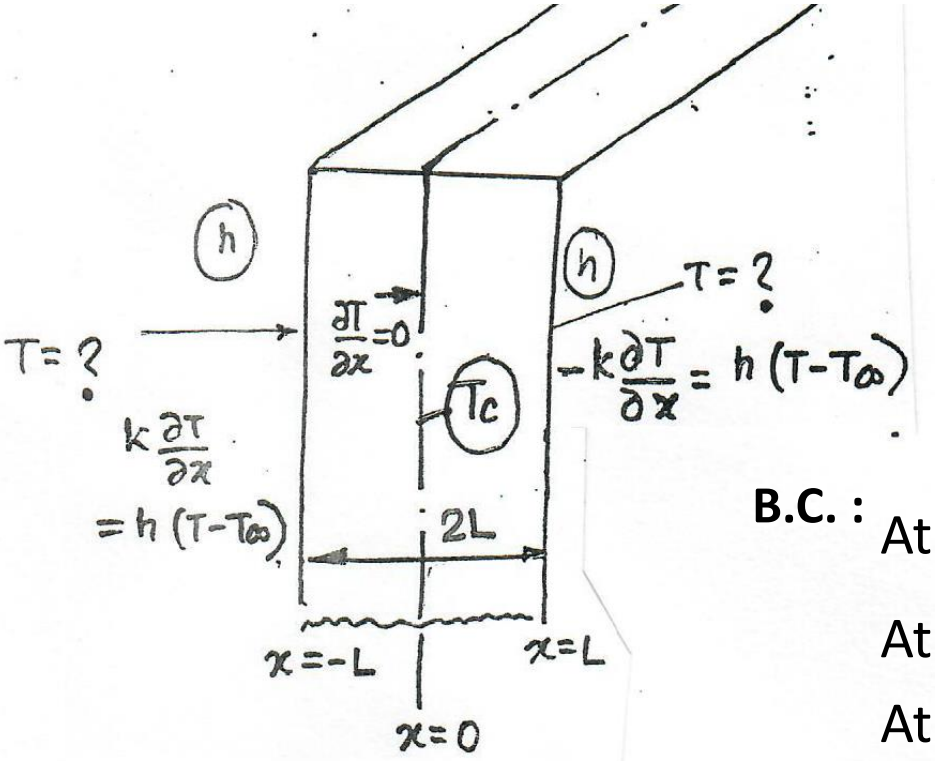
Heat and Mass transfer

UNSTEADY STATE CONDUCTION IN AN INFINITE SLAB

TRANSIENT CONDUCTION IN RECTANGULAR GEOMETRY WITH CONVECTIVE BOUNDARY CONDITION



- Local ambience is at $T = T_\infty$.
- Temperature at the surface is not known.
- h is known.

G.E. : $\dfrac{\partial T}{\partial t} = \alpha \dfrac{\partial^2 T}{\partial x^2}$

B.C. : At $x = 0, \dfrac{\partial T}{\partial x} = 0$, for $t > 0$

At $x = L, -k\dfrac{\partial T}{\partial x} = h(T - T_\infty)$, for $t > 0$

At $t = 0, T = T_o$, for $0 \le x \le L$

Put, $\theta = (T - T_\infty)$,

G.E. : $\dfrac{\partial \theta}{\partial t} = \alpha \dfrac{\partial^2 \theta}{\partial x^2}$

B.C. : At $x = 0, \dfrac{\partial \theta}{\partial x} = 0$, for $t > 0$

At $x = L, -k\dfrac{\partial \theta}{\partial x} = h\theta$, for $t > 0$

At $t = 0, \theta = \theta_o$, for $0 \le x \le L$

Let, $\theta = X(x)\Gamma(t)$, and the governing equation will be

$$\alpha X''\Gamma = X\,\dot{\Gamma}$$

$$\text{or,} \quad \frac{\dot{\Gamma}}{\alpha\Gamma} = \frac{X''}{X} = \text{constant} = \pm\lambda^{2}$$

As it was shown in the earlier problem, the constant should be $-\lambda^{2}$ and the solution for Γ , is given by

$$\Gamma = c_1 e^{-\alpha\lambda^2 t}$$

Where, c_1 is any arbitrary constant.

Now R.H.S. can be written as: $\frac{X''}{X} = -\lambda^2$

Solution is, $X = c_2 \cos\lambda x + c_3 \sin\lambda x$, where c_2 and c_3 are arbitrary constants.

Now, $\boldsymbol{\theta = X(x)\Gamma(t) = e^{-\alpha\lambda^2 t}[A\cos\lambda x + B\sin\lambda x]}$ **(1)**

$$\frac{\partial\theta}{\partial x} = e^{-\alpha\lambda^2 t}[-A\lambda\sin\lambda x + B\lambda\cos\lambda x]$$

At $x = 0$, $\frac{\partial \theta}{\partial x} = 0$ for $t > 0$ } Substituting this boundary condition in the above equation yields, $B = 0$

Therefore, $\boldsymbol{\theta = e^{-\alpha\lambda^2 t} A \cos \lambda x}$

$$-k\frac{\partial \theta}{\partial x} = +k\lambda A \sin \lambda x \, e^{-\alpha\lambda^2 t}$$

$$-k\frac{\partial \theta}{\partial x}\Big|_{x=L} = h\theta$$

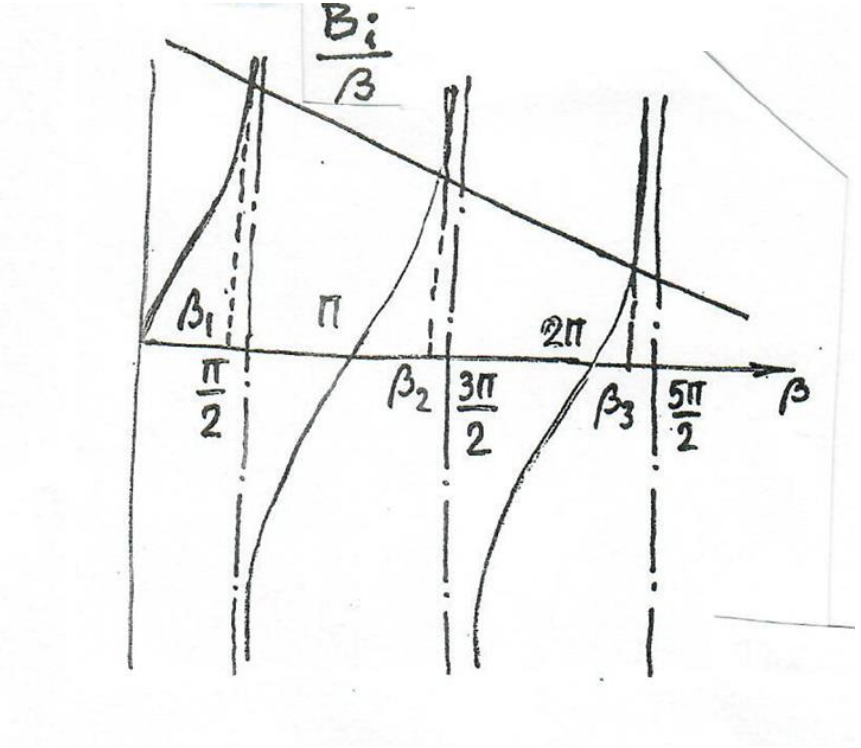
$$k\lambda A \sin \lambda L \, e^{-\alpha\lambda^2 t} = h e^{-\alpha\lambda^2 t} A \cos \lambda L$$

or, $\lambda \tan \lambda L = \frac{h}{k}$

or, $\lambda L \tan \lambda L = \frac{hL}{k} = Bi$ **(2)**

where, ***Bi*** is ***Biot Number.***

Or, $\beta \tan \beta = Bi$, where $\beta = \lambda L$ **(3)**



This equation is a transcendental equation and it has an infinite number of roots, the eigen values β, viz, $\beta_1, \beta_2, \beta_3, \ldots\ldots\ldots, \beta_n$ are shown graphically (points of intersection). These are also available in a tabular form.

Once $\beta_n = \lambda_n L$, is known λ_n is known, and we get the solution as,

$$\theta = \sum_{n=1}^{\infty} A_n \cos\lambda_n x e^{-\alpha\lambda_n^2 t}$$

The initial condition is : at $t = 0,\ \theta = \theta_o$

$$\theta_o = \sum_{n=1}^{\infty} A_n \cos\lambda_n x$$

$$\int_0^L \theta_o cos\lambda_m x \, \mathrm{dx} = \int_0^L \{A_n cos\lambda_n x\} \, cos\lambda_m x \ \ dx$$

For $m = n$,

$$\int_0^L \theta_o \cos\lambda_n x \, dx = \int_0^L \Sigma A_n \cos^2\lambda_n x \ \ dx$$

or,

$$\left[\frac{\theta_o \sin\lambda_n x}{\lambda_n}\right]_o^L = \frac{A_n}{2}\int_0^L [1 + \cos 2\lambda_n x] dx$$

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$$\left[\frac{\theta_o \sin \lambda_n L}{\lambda_n}\right] = \frac{A_n}{2}\left[x + \frac{\sin 2\lambda_n x}{2\lambda_n}\right]_0^L$$

$$A_n = \frac{4\,\theta_o \sin \lambda_n L}{\left[2\lambda_n L + \sin 2\lambda_n L\right]}$$

$$A_n = \frac{4\theta_o \sin \beta_n}{2\beta_n + \sin 2\beta_n} \qquad \textbf{(4)}$$

Where, $\beta_n = \lambda_n L$

Now,

$$\theta = \Sigma_{n=1}^{\infty} \frac{4\theta_o \sin\beta_n}{2\beta_n + \sin 2\beta_n} \cos\left(\frac{\beta_n x}{L}\right) e^{-\alpha\left(\frac{\beta_n}{L}\right)^2 t} \qquad \textbf{(5)}$$

We know, $\beta_n = \lambda_n L$

We also observe that, $\frac{\theta}{\theta_o} = f\left(\lambda_n L,\ \frac{\alpha t}{L^2}, \frac{x}{L}\right)$

Where, $\frac{\alpha t}{L^2} = \boldsymbol{Fo}$, **(*Fourier Number)***

From **(2)**, we have also seen that,

$$\lambda L = f\left(\frac{hL}{k}\right) = f(Bi)$$

We may say that the dimensionless temperature distribution is a function of Bi, Fo and $\frac{x}{L}$

Heisler showed that the infinite series in equation (5) converges rapidly and that for values of $\frac{\alpha t}{L^2}$ greater than 0.2, all terms after the first can be neglected.

Thus one needs to know only the first eigen value in the equation(5).Using this fact, equation(5) reduces to

$$\frac{\theta}{\theta_o} = \frac{4\sin\lambda_1 L}{2\lambda_1 L + \sin^2(2\lambda_1 L)} \cos\left(\lambda_1 \mathrm{L} \cdot \frac{\mathrm{x}}{\mathrm{L}}\right) \mathrm{e}^{-(\lambda_1 \mathrm{L})^2 \frac{\alpha \mathrm{t}}{\mathrm{L}^2}} \qquad \textbf{(6)}$$

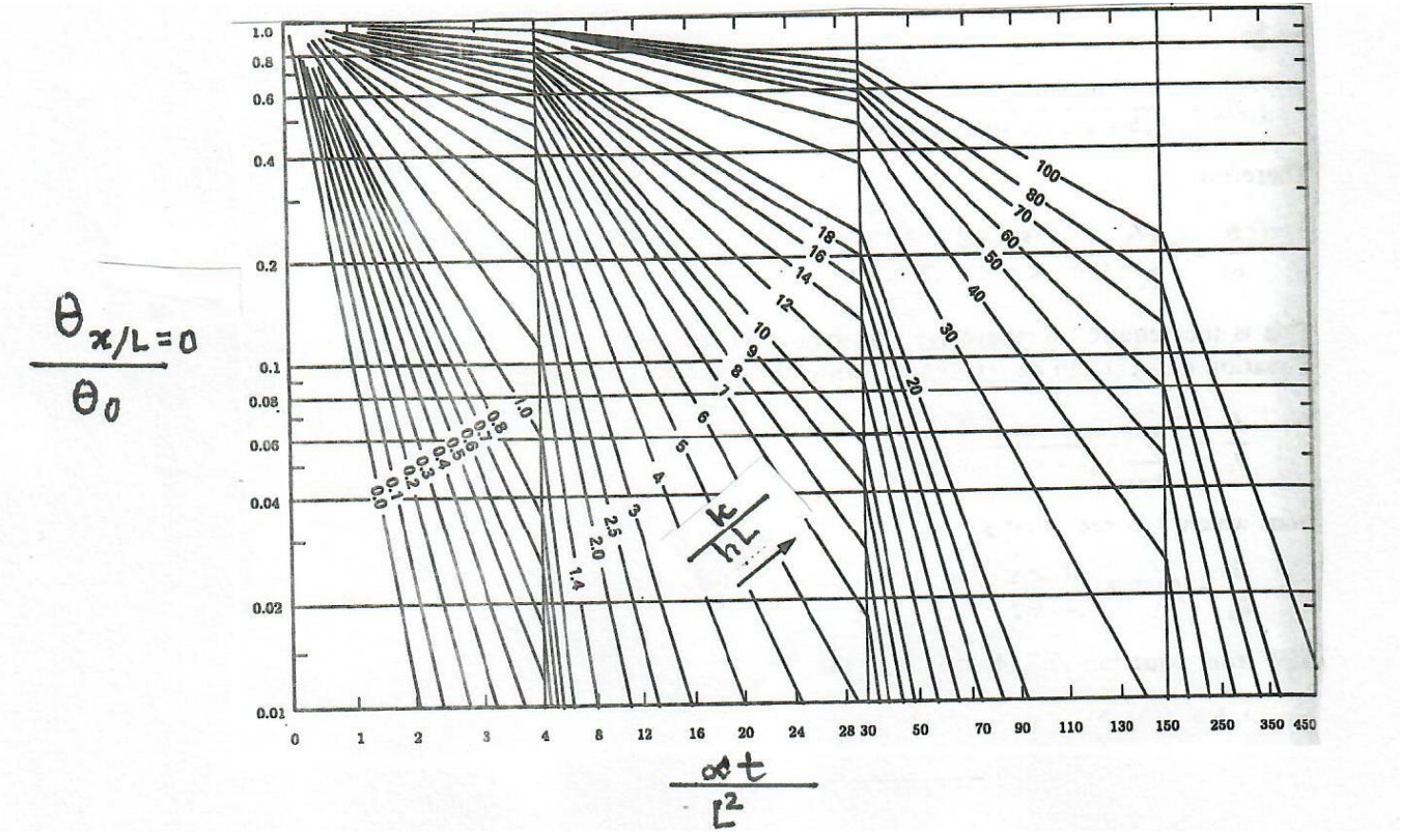
Setting, $\frac{x}{L} = 0$, we have the dimensionless temperature at the centre-line given by

$$\frac{\theta_{\mathrm{x/L=0}}}{\theta_o} = \frac{4\sin\lambda_1 L}{2\lambda_1 L + \sin^2(2\lambda_1 L)} \mathrm{e}^{-(\lambda_1 \mathrm{L})^2 \frac{\alpha \mathrm{t}}{\mathrm{L}^2}} \qquad \textbf{(7)}$$

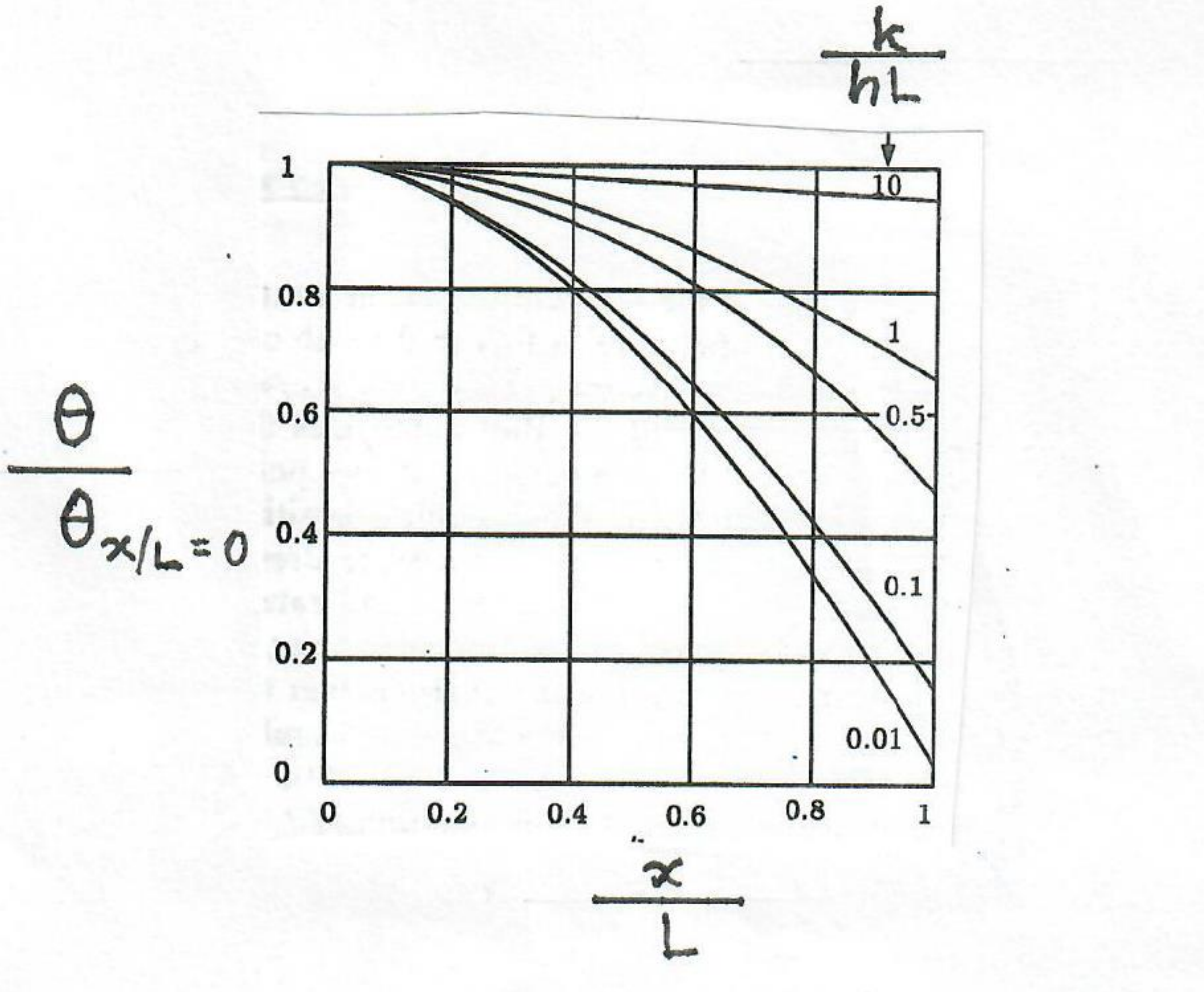
Dividing equation (6) by equation(7), we have

$$\frac{\theta}{\theta_{x/L=0}} = \cos\left(\lambda_1 L \cdot \frac{x}{L}\right) \qquad \textbf{(8)}$$

- Thus the ratio of the temperature at any point in the slab to the temperature at the centre-line **is** <u>independent of time</u>.
- Consequently the complete temperature history in the slab can be drawn in two charts, called **Heisler Charts** (named after M.P. Heisler).
- In one chart, the dimensionless centre-line temperature $(\theta_{x/L=0})$ θ_o is plotted against the Fourier number Fo , with the inverse of the **Biot Number** $\boldsymbol{Bi}$ as the parameter.



- In the second chart, the dimensionless ratio of the temperature at any point in the slab and the centre-line temperature is plotted against the dimensionless position.



The corresponding analytical solution for the one- dimensional unsteady state temperature distributions in a long solid cylinder or a solid sphere, initially at a uniform temperature and subsequently subjected to a different ambient temperatures, can be obtained in a similar fashion. Heisler charts are also available to determine the temperature distribution graphically.

Thank you